THE HAUSDORFF DIMENSION OF THE SAMPLE PATH OF A SUBORDINATOR

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ABSTRACT

The Hausdorff dimension of the range of an arbitrary subordinator is exactly determined in terms of the rate of linear drift and the Levy measure of the subordinator. This generalizes the result of Blumenthal and Getoor: that for a stable subordinator of index σ, the dimension of the range is σ.

1. Introduction. Let $T(s)$, $s \geqq 0$, be a subordinator, i.e. a real-valued random process having stationary, independent increments and a.s. increasing sample paths, defined on a probability space $(\Omega, \mathscr{F}, P)$. We may assume that $T(0) = 0$ a.s. and that the paths of $T(s)$ are right continuous [2]. Such processes are characterized by the Laplace transform $E(e^{-\lambda T(s)})$ which, for subordinators, takes the form $E(e^{-\lambda T(s)}) = e^{-sg(\lambda)}$, where $g(\lambda) = \alpha\lambda + \int_0^\infty (1 - e^{-\lambda y}) n(dy)$ is the *subordinator exponent*. The constant $\alpha \geqq 0$ is the *rate of linear drift* and the measure n is the *Lévy measure* of T; cf. [5, p. 31–32]. The purpose of this paper is to determine the Hausdorff dimension of the range of $T(s)$ in terms of the parameters α and n.

Define $H(x) = n(x, \infty)$. From the finiteness of $g(\lambda)$ it follows that $H(x)$ is finite on $(0, \infty)$ and $\int_0^1 H(x)\,dx < \infty$. Further, H is nonnegative, nonincreasing, and right continuous on $(0, \infty)$. The main result is the following:

THEOREM. *Let T be the subordinator with exponent $g(\lambda)$, and let $Q = Q(\omega)$ be the range of $T(s,\omega)$, $s \geqq 0$. Then*

(i) if $\alpha > 0$, $\dim Q = 1$ a.s.

(ii) if $\alpha = 0$, $\dim Q = \sigma$ a.s.

where $\sigma = \sup\{\gamma \leqq 1 : x^{\gamma - 1} \int_0^x H(y)dy \to \infty$ as $x \to 0\}$, *and* $\dim Q$ *denotes the Hausdorff dimension of* Q.

This generalizes a result of Blumenthal and Getoor [1] on stable subordinators, and improves a further result of theirs [2] in the general case. In [2] it was shown that $\dim\{T(s): 0 \leqq s \leqq 1\} \geqq \sigma'$, where $\sigma' = \sup\{\gamma \leqq 1 : \lambda^{-\gamma} g(\lambda) \to \infty$ as $\lambda \to \infty\}$. The inequalities

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$$e^{-1}\lambda^{1-\gamma}\int_0^{1/\lambda} H(y)\,dy \leqq \lambda^{-\gamma} g(\lambda) = \lambda^{1-\gamma}\int_0^{\infty} e^{-\lambda y} H(y)\,dy$$
$$\leqq (1+e^{-1})\lambda^{1-\gamma}\int_0^{1/\lambda} H(y)\,dy$$

show that $\sigma = \sigma'$, and thus Blumenthal-Getoor's lower bound is actually the dimension. For the sake of completeness, we will give a new proof of the fact that $\dim Q \geqq \sigma$. Finally let us mention that in effect we have determined the dimension of the zero set for a large class of Markov processes, including those covered by the theory of local times as in [3].

In §2 we dispose of two easy cases which arise, and then outline the method to be used in the remaining cases. §3 contains a brief description of semilinear processes — the main tool in the proof of the theorem — and in §4 we complete the proof.

2. Preliminaries. We refer the reader to [5, p. 53] for the definition of Hausdorff β-dimensional measure and Hausdorff dimension for linear point sets.

Let us first prove the theorem in two special cases.

Case 1. $H(0+) < \infty, \alpha > 0$. Using the Lévy decomposition of $T(s)$ into a linear part plus a saltus part [5, p. 31], it is easy to see that the graph of $T(s)$ consists of a countable collection of line segments of slope α, and thus Q has positive Lebesgue measure. Therefore $\dim Q = 1$. The details here and in Case 2 are left to the reader.

Case 2. $H(0+) < \infty$, $\alpha = 0$. In this case the graph of T is a countable collection of horizontal line segments, so Q is countable and $\dim Q = 0$.

For the rest of the paper we assume that $H(0+) = \infty$. It is shown in [4, p. 63] that the subordinator T with exponent $g(\lambda)$ may be regarded as the inverse local time at zero of the semilinear strong Markov process x_t with characteristic $\{\alpha, H(x)\}$ (see §3 below). Roughly speaking, if we define the random function $\xi_t = t - \sup(Q \cap [0,t])$ then there is a strongly Markov process $X = (x_t, \mathcal{N}_t, P^x)$, $x \geqq 0$, such that x_t under P^0 is equivalent to ξ_t under P. Thus, if $Z = \{t : x_t = 0\}$ is the zero set of x_t, we have in effect $Z = \bar{Q}$. Since $\bar{Q}\backslash Q$ is countable, $\dim Q = \dim Z$. We therefore study x_t as the primary process. This connection between subordinators and semilinear processes is similar to that between stable processes of index γ, $1 < \gamma \leqq 2$, and stable subordinators of index $\beta = 1 - 1/\gamma$. The latter was exploited (though in the opposite direction) in [7].

3. Semilinear Processes. These processes were first studied in [6] under the name Markov random sets, and later in [4], from which the following is obtained. A Markov process $X = (x_t, \mathcal{N}_t, P^x)$, with state space E an interval $[0, a)$, $a \leqq \infty$, is *semilinear* if its trajectories have the following shape: let Z be the zero set of x_t and let τ be the hitting time of $\{0\}$, $\tau = \inf(t > 0 : x_t = 0)$. Z is assumed to be closed. Then

$$x_t = t - \sup(Z \cap [0,t]) \qquad P^0 - \text{a.s.}$$

$$x_t = \begin{cases} x+t & t < \tau \\ t - \sup(Z \cap [0,t]) & t \geqq \tau \end{cases} \qquad P^x - \text{a.s.},\ 0 < x \in E.$$

When $H(0+) = \infty$, $\tau = 0$ P^0 – a.s., so x_t is well-defiined. Each trajectory is thus an irregular saw tooth with infinitely many teeth in any time interval $[0,t]$, and the value of x_t equals the time elapsed since the last zero before time t.

The *characteristic* of the semilinear process x_t is the pair $\{\beta, h(x)\}$ determined (up to a positive multiplicative constant) by

$$P^x(\tau > t) = \frac{h(x+t)}{h(x)}$$

$$(0 < x \in E)$$

$$E^0(\tau(x)) = \frac{\beta + \int_0^x h(y)\,dy}{h(x)}$$

where $\tau(x)$ is the hitting time of $\{x\}$, $x \in E$. The constant β is nonnegative, and $h(x)$ is a nonnegative, nonincreasing, right continuous function on E such that $\int_0^x h(y)\,dy < \infty$ if $x \in E$. Conversely, for each pair $\{\beta, h(x)\}$, $\beta \geqq 0$, $h(x)$ as described, and with $h(0+) = \infty$, there is a unique strongly Markov semilinear process x_t with characteristic $\{\beta, h(x)\}$. For any constant $c > 0$, the pairs $\{\beta, h(x)\}$ and $\{c\beta, ch(x)\}$ determine equivalent processes. With no loss of generality we can assume that $E = [0,\infty)$, $h(x) > 0$ for $x > 0$, and $h(1) = 1$.

LEMMA 1. [4, p. 46]. *For* $x > 0$,

$$E^0(e^{-\lambda\tau(x)}) = \frac{e^{-\lambda x}}{1 + \dfrac{\beta\lambda}{h(x)} + \dfrac{1}{h(x)} \displaystyle\int_0^x (e^{-\lambda z} - 1)\,dh(z)}.$$

It is shown in [4, ch. 5] that, if x_t is a strongly Markov semi-linear process with characteristic $\{\beta, h(x)\}$, where $h(0+) = \infty$, then x_t has a local time A_t at zero. A_t may be characterized as the unique continuous additive functional of x_t whose set of increase points coincides with the zero set Z of x_t.

The inverse local time of x_t is the process $T(s) = \inf\{t : A_t > s\}$. The process $T(s)$ has right continuous, a.s. increasing sample paths, and $T(0) = 0$ P^0 – a.s. Finally, under the measure P^0, T is a subordinator with exponent

$$\beta\lambda + \int_0^\infty (1 - e^{-\lambda y})\,m(dy)$$

with m the measure on $(0,\infty)$ determined by $h(x) = m(x,\infty)$; see [4, p. 63]. The range Q of T satisfies $\bar{Q} = Z$. Thus, if we start with the semilinear process x_t with characteristic $\{\alpha, H(x)\}$, we get as the inverse local time the subordinator T with drift α and Lévy measure n as in §1. For the remainder of the paper x_t will be strongly Markov semilinear, with characteristic $\{\alpha, H(x)\}, H(x) > 0$ for $x > 0$, $H(1) = 1$, $H(0+) = \infty$, and A_t will be the local time at zero for x_t.

4. **Proof of the theorem for** $H(0+) = \infty$. Suppose first that $\alpha > 0$. It is known [4, p. 86] that $m(Z \cap [0,t]) = A_t$, where m is Lebesgue measure. But $H(0+) = \infty$ implies $T(s) > 0$ for $s > 0$, hence $A_t > 0$ for $t > 0$. Thus Z has positive Lebesgue measure and $\dim Z = 1$.

Suppose now that $\alpha = 0$.

LEMMA 2. *If* $\gamma < \sigma$, *then* $\lim_{h \downarrow 0} h^{-\gamma}(A_{t+h} - A_t) = 0$ P^0 – a.s. *for each* $t \geqq 0$.

Set $G(t) = E^0 A_t$. Then $G(t)$ is a nondecreasing continuous function which determines a measure on $[0,\infty)$ which we denote again by G. The Laplace-Stieltjes transform of G is given by

$$\hat{G}(\lambda) = \int_0^\infty e^{-\lambda t} G(dt) = \frac{1}{g(\lambda)}$$

where $g(\lambda)$ is the exponent of T, cf. [4, p. 63]. Clearly $e^{-1}G(1/\lambda) \leqq 1/g(\lambda)$. Moreover, $G(t+h) - G(t) \leqq G(h)$ for $t \geqq 0$, $h \geqq 0$ [4, p. 68].

Now, for t, h fixed,

$$E^0 \int_0^h r^{-\gamma} d(A_{t+r} - A_t) \leqq h^{-\gamma} E^0(A_{t+h} - A_t) + \gamma \int_0^h r^{-\gamma-1} E^0(A_{t+r} - A_t)\, dr$$

$$\leqq h^{-\gamma} G(h) + \gamma \int_0^h r^{-\gamma-1}\, G(r)\, dr.$$

Let $\gamma < \sigma$ be arbitrary, and choose β, $\gamma < \beta < \sigma$. Since $\sigma = \sigma'$ (see §1), we have $g(\lambda) > \lambda^\beta$ for sufficiently large λ, hence $G(r) = O(r^\beta)$ as $r \to 0$. The right member of the above inequality is therefore finite, and we conclude

$$h^{-\gamma}(A_{t+} - A_t) \leqq \int_0^h r^{-\gamma} d(A_{t+r} - A_t) \to 0 \quad (h \downarrow 0) \quad P^0 - \text{a.s.},$$

which proves Lemma 2.

By an argument like that following Lemma 4 of [7] we see that the γ-dimensional measure of $Z \cap [0,t]$ is positive for every t, thus $\dim Z \geqq \gamma$. Since $\gamma < \sigma$ was arbitrary, we have proven: $\dim Z \geqq \sigma$.

It remains to prove the opposite inequality. Because of Lemma 1 it is clear that $\infty > \tau(n) \uparrow \infty$ P^0 – a.s., so it suffices to show that $\dim(Z \cap [0,\tau(n)]) \leqq \sigma$ for $n = 1, 2, \cdots$. We prove this for $n = 1$, leaving the rest for the reader. (Recall $\tau(x)$ is the hitting time of $\{x\}$.)

Fix $0 < \varepsilon < 1$ and define $\tau_k = \tau_k(\varepsilon)$ to be the kth hitting time of $\{\varepsilon\}$. Let $\sigma_k = \sigma_k(\varepsilon)$ be the first zero of x_t after $\tau_k(\varepsilon)$. More precisely,

$$\tau_1 = \tau(\varepsilon) = \inf(t > 0 : x_t = \varepsilon), \ \tau_k = \tau_{k-1} + \tau_1(\theta_{\tau_{k-1}}),$$

$$\sigma_0 = 0, \ \sigma_k = \tau_k + \tau(\theta_{\tau_k}), \ k \geqq 1$$

where θ_t is the shift operator of the process x_t and τ is the hitting time of $\{0\}$. Each of the times τ, τ_k, σ_k is a stopping time of the process x_t.

Define $d_\varepsilon(t)$ as the number of downcrossings made by x_t from ε down to 0 during the time $(0, t]$. Notice that $d_\varepsilon(t) \geqq k$ iff $\sigma_k \leqq t$, under P^0.

Let $I_k = I_k(\varepsilon) = [\sigma_{k-1}, \tau_k - \varepsilon]$, so I_k is a "random interval" depending on the sample path x_t. We allow $1 \leqq k \leqq d_\varepsilon(\tau(1)) + 1$. Thus the intervals I_k form a covering of $Z \cap [0, \tau(1)]$ by closed, disjoint intervals. Since $\tau(1) < \infty$ $P^0 -$ a.s. (Lemma 1), $Z \cap [0, \tau(1)]$ is a.s. compact. Also, $Z \cap [0, \tau(1)]$ has Lebesgue measure zero [4, p. 104], hence

$$\max_{1 \leqq k \leqq K\varepsilon} |I_k| \to 0 \ P^0 - \text{a.s. as } \varepsilon \to 0;$$

here $K_\varepsilon \equiv d_\varepsilon(\tau(1)) + 1$, and $|I|$ denotes the length of the interval I.

Let $0 \leqq \beta \leqq 1$, and put $\Delta_{\varepsilon,\beta} = \sum_{j=1}^{K_\varepsilon} |I_k|^\beta$. If $E^0(\Delta_{\varepsilon,\beta}) \to 0$ as $\varepsilon \to 0$ through an appropriate sequence ε_n then $\Delta_{\varepsilon_{n_k},\beta} \to 0$ $P^0 -$ a.s. for some subsequence ε_{n_k}. It follows from this that the β-dimensional measure of $Z \cap [0, \tau(1)]$ is zero, and therefore $\dim(Z \cap [0, \tau(1)]) \leqq \beta$. Thus we must show that, for every $\beta > \sigma$, $E^0(\Delta_{\varepsilon_n,\beta}) \to 0$ for a suitable sequence ε_n (which may depend on β). We can assume $\sigma < 1$; otherwise we would have $\dim Z = 1$ automatically.

Lemma 3. *For fixed* $\varepsilon, \beta, E^0(\Delta_{\varepsilon,\beta}) = H(\varepsilon) \int_0^\infty t^\beta d\mu_\varepsilon(t)$, *where* μ_ε *is the measure on* $[0, \infty)$ *with Laplace-Stieltjes transform*

$$\hat{\mu}_\varepsilon(\lambda) = 1/\{1 + (H(\varepsilon))^{-1} \int_0^\varepsilon (e^{-\lambda z} - 1)\, dH(z)\}.$$

Notice first that $|I_j| = \tau_j - \varepsilon - \sigma_{j-1} = \tau_1(\theta_{\sigma_{j-1}}) - \varepsilon$, since $\tau_j = \sigma_{j-1} + \tau_1(\theta_{\sigma_{j-1}})$. Now

$$\begin{aligned} E^0(\Delta_{\varepsilon,\beta}) &= \sum_{k=1}^\infty E^0\left(\sum_{j=1}^k (\tau_1(\theta_{\sigma_{j-1}}) - \varepsilon)^\beta ; d_\varepsilon(\tau(1)) = k - 1\right) \\ &= \sum_{j=1}^\infty \sum_{k \geqq j} E^0((\tau_1(\theta_{\sigma_{j-1}}) - \varepsilon)^\beta ; d_\varepsilon(\tau(1)) = k - 1) \\ &= \sum_{j=0}^\infty E^0((\tau_1(\theta_{\sigma_j}) - \varepsilon)^\beta ; \sigma_j \leqq \tau(1)) \\ &= E^0(\tau_1 - \varepsilon)^\beta \sum_{j=0}^\infty P^0(\sigma_j \leqq \tau(1)). \end{aligned}$$

The last line follows by the strong Markov property applied to σ_j, and the fact that $x_{\sigma_j} = 0$.

Since

$$P^0(\sigma_j \leqq \tau(1)) = P^0(\sigma_1 + \sigma_{j-1}(\theta_{\sigma_1}) \leqq \sigma_1 + \tau(1, \theta_{\sigma_1}),\ \sigma_1 \leqq \tau(1)) = P^0(\sigma_{j-1} \leqq \tau(1)) P^0(\sigma_1 \leqq \tau(1))$$

by the strong Markov property, we have $P^0(\sigma_j \leqq \tau(1)) = (P^0(\sigma_1 \leqq \tau(1)))^j$. But $P^0(\sigma_1 \leqq \tau(1)) = P^0(\tau(\theta_{\tau_1}) \leqq 1 - \varepsilon) = 1 - H(1)/H(\varepsilon)$ (see §3), so that the sum of the above series is $H(\varepsilon)$.

Let Q_ε be the distribution function of $\tau_1 = \tau(\varepsilon)$. Because of the shape of the trajectory x_t, it is clear that $\tau_1 \geqq \varepsilon$ P^0 – a.s. Thus

$$E^0(\tau_1 - \varepsilon)^\beta = \int_\varepsilon^\infty (t - \varepsilon)^\beta \, dQ_\varepsilon(t) = \int_0^\infty t^\beta \, d\mu_\varepsilon(t),$$

where $d\mu_\varepsilon(t) = dQ_\varepsilon(t + \varepsilon)$. Lemma 3 now follows from Lemma 1.

Write $\nu_\varepsilon = H(\varepsilon)\mu_\varepsilon$. By Lemma 3, if $\beta < 1$,

$$\begin{aligned} E^0(\Delta_{\varepsilon,\beta}) &= \int_0^\infty t^\beta d\nu_\varepsilon(t) \\ &= \frac{1}{\Gamma(1-\beta)} \int_0^\infty y^{-\beta} \int_0^\infty t e^{-ty} \, d\nu_\varepsilon(t) \, dy \\ &= \frac{1}{\Gamma(1-\beta)} \int_0^\infty y^{-\beta} \left(-\frac{d}{dy} \hat{\nu}_\varepsilon(y) \right) dy. \end{aligned}$$

The integral is finite since $E^0(\tau_1) < \infty$.

Fix $1 > \beta > \sigma$, and choose $\delta > 0$ such that $\beta - \delta > \sigma$. By definition of σ, there is a sequence $\varepsilon_n \to 0$ such that

$$\varepsilon_n^{\beta - \delta - 1} \int_0^{\varepsilon_n} H(y) \, dy \leqq M < \infty.$$

Thus $\varepsilon_n^{\beta-1} \int_0^{\varepsilon_n} H(y) \, dy \to 0$. We show that $E^0(\Delta_{\varepsilon_n,\beta}) \to 0$.

Now, for $0 < \varepsilon < 1$,

$$\int_0^\infty y^{-\beta} \left(-\frac{d}{dy} \hat{\nu}_\varepsilon \right) dy = \int_0^1 + \int_1^{1/\varepsilon} + \int_{1/\varepsilon}^\infty \equiv A + B + C;$$

$$A \leqq \int_0^1 y^{-\beta} \int_0^\varepsilon z e^{-yz} dH_1(z) \, dy \leqq \frac{1}{1-\beta} \int_0^\varepsilon z \, dH_1(z),$$

where $H_1(z) = -H(z)$ and

$$-\frac{d}{dy} \hat{\nu}_\varepsilon = \frac{\int_0^\varepsilon z e^{-yz} dH_1(z)}{(1 + (H(\varepsilon))^{-1} \int_0^\varepsilon (e^{-yz} - 1) \, dH(z))^2}.$$

Thus $A \to 0$ as $\varepsilon \to 0$. As for B, we have

$$B \leqq \int_1^{1/\varepsilon} y^{-\beta} \int_0^{\varepsilon} e^{-yz}(1 - yz) H(z)\, dz\, dy \leqq \frac{1}{1-\beta} \varepsilon^{\beta-1} \int_0^{\varepsilon} H(z)\, dz.$$

By our choice of ε_n, $B \to 0$ as $\varepsilon \to 0$ through the sequence ε_n. Finally, integrating C by parts, we obtain

$$C \leqq \varepsilon^{\beta} \hat{v}_{\varepsilon} \left(\frac{1}{\varepsilon}\right) \leqq \varepsilon^{\beta} H(\varepsilon) \leqq \varepsilon^{\beta-1} \int_0^{\varepsilon} H(y)\, dy.$$

Therefore $C \to 0$ as $\varepsilon \to 0$ through the sequence ε_n. Thus $E^0(\Delta_{\varepsilon_n,\beta}) \to 0$, and the theorem is proven.

A similar proof yields, for example, the following:

COROLLARY. *Let $T(s)$ be a stable subordinator of index σ* (i.e. $g(\lambda) = \lambda^{\sigma}$). *Then the σ-dimensional measure of the range of T is finite.*

In fact, it follows from [7] that "finite" may be replaced by "zero". We have included this result merely to indicate some possibilities in computing Hausdorff measures for subordinators.

REFERENCES

1. R. M. Blumenthal and R. K. Getoor, *Some theorems on stable processes*, Trans. Amer. Math. Soc., **95**, (1960), 263–273.

2. ———, *Sample functions of stochastic processes with stationary independent increments*, J. Math. Mech., **10** (1961), 493–516.

3. ———, *Local times for Markov processes*, Z. Wahr., **3** (1964), 50–74.

4. J. Horowitz, *Semilinear Markov processes*, Thesis, University of Michigan, 1967.

5. K. Ito and H. P. McKean, Jr., *Diffusion Processes and their Sample Paths*, Academic Press, New York, 1965.

6. N. V. Krylov and A. A. Jushkevitch, *Markov random sets*, Trudy, Moscow Math. Soc., **13** (1965), 114–135 (in Russian).

7. S. J. Taylor and J. G. Wendel, *The exact Hausdorff measure of the zero set of a stable process*, Z. Wahr., **6** (1966), 170–180.

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